Announcements

1) 1^{st} homework will appear either tonight or tomorrow on CTools.

Counterexamples:

If you think a statement is true, you need to prove it for all cases it is applicable to.

If you think a statement is **false**, all you need to do is construct a single counterexample.

I.E The statement is

Every natural number is either prime or has a unique prime divisor.

This statement is false since $n=6$ is a counterexample $(6 = 2 \cdot 3)$

Basic Set Theory and Notation

We will not define a set and regard it as axiomatically given

We shall obey the Zermelo-Frankel (ZF) axioms of set theory, though we may employ the axiom of choice (ZFC)

Q: What is an axiom?

An axiom is a statement that we will assume to be true. It is beyond the realm of proof.

Let S, T be sets.

$S \cap T = \{x \mid x \in S \text{ and } x \in T\}$

(intersection of S and T)

$S \cup T = \{x \mid x \in S \text{ or } x \in T\}$

(union of S and T)

$\emptyset$ = the set with no elements

$= \{x \mid x \neq x\}$.

Examples:

Let $S = \{1, 3, 5, 6\}$

$T = \{2, 5, 7\}$

Then $S \cap T = \{5\}$

$S \cup T = \{1, 2, 3, 5, 6, 7\}$

If we take $V = \{2, 4\}$

Then $S \cap V = \emptyset$

(no elements in common)

Infinite Intersections / Unions

Let D be an infinite set, $\{S_d\}_{d\in D}$ a collection of sets indexed by D. We define

$$\bigcap_{d\in D} S_d = \{x \mid x\in S_d \ \forall d\in D\}$$

$$\bigcup_{d\in D} S_d = \{x \mid \exists d\in D,\ x\in S_d\}$$

We will usually take $D = \mathbb{N}$. In that case, we write

$$\bigcap_{n \in \mathbb{N}} S_n \quad \text{as} \quad \bigcap_{n=1}^{\infty} S_n$$

and

$$\bigcup_{n \in \mathbb{N}} S_n \quad \text{as} \quad \bigcup_{n=1}^{\infty} S_n .$$

Example:

$$S_n = \{k \in \mathbb{N} : k > 2^n\}$$

$$S_1 = \{k \in \mathbb{N} : k > 2\}$$

$$= \{3, 4, 5, \ldots\}$$

$$S_2 = \{k \in \mathbb{N} : k > 4\}$$

$$= \{5, 6, 7, \ldots\}$$

What is

$$\bigcup_{n=1}^{\infty} S_n \text{ ?}$$

$$\bigcup_{n=1}^{\infty} S_n = S_1$$ since

S_n is contained in S_{n-1}.

So $$\bigcup_{n=1}^{\infty} S_n = \{ k > 2 \mid k \in \mathbb{N} \}$$

What is

$$\bigcap_{n=1}^{\infty} S_n ?$$

$$\bigcap_{n=1}^{\infty} S_n = \phi$$ since

$$\bigcap_{n=1}^{m} S_n = S_m .$$ So elements in $\bigcap_{n=1}^{\infty} S_n$ are numbers $k \in \mathbb{N}$ such that $k > 2^n$ for every $n \in \mathbb{N}$.

There are no such natural numbers since if $k \in \mathbb{N}$, $k < 2^k$. Therefore,

$$\bigcap_{n=1}^{\infty} S_n = \emptyset.$$

$\boxed{S \subseteq T}$ if every $x \in S$ is an element of T

<u>Note!</u> $S = T$ is allowed (if and only if $S \subseteq T$ and $T \subseteq S$)

$$S^c = \{x \mid x \notin S\}$$

$S \cap T = \emptyset$ if and only if $T \subseteq S^c$.

De Morgan's Laws: Let S, T be sets,

i) $(S \cup T)^c = S^c \cap T^c$

ii) $(S \cap T)^c = S^c \cup T^c$

Proof: i) We will show $(S \cup T)^c \subseteq S^c \cap T^c$ and $S^c \cap T^c \subseteq (S \cup T)^c$

Let $x \in (S \cup T)^c$.

This means

$x \notin S \cup T$.

If $x \notin S \cup T$, then

x is neither in S nor in T.

This is equivalent to saying

$x \in S^c$ and $x \in T^c$, so

$x \in S^c \cap T^c$.

Therefore, $(S \cup T)^c \subseteq S^c \cap T^c$

All these steps are reversible, so starting with $x \in S^c \cap T^c$, we obtain $x \in (S \cup T)^c$ by running the previous argument backwards.

(ii) Similar.

$\square$

Functions Between Sets

(mainly concerned with $S = T = \mathbb{R}$, or subsets thereof)

Definition: Let S and T be sets. A function f from S to T is a rule that assigns a unique element of T to each $s \in S$. **Notation:** $f : S \to T$.

Example 2: Let $S = T = \mathbb{R}$

Define $f: \mathbb{R} \to \mathbb{R}$ by

$$f(x) = x^2.$$

Note: f does not map onto all of $\mathbb{R}$! This is ok for the notation.

Example 4: (Dirichlet)

$S = T = \mathbb{R}$

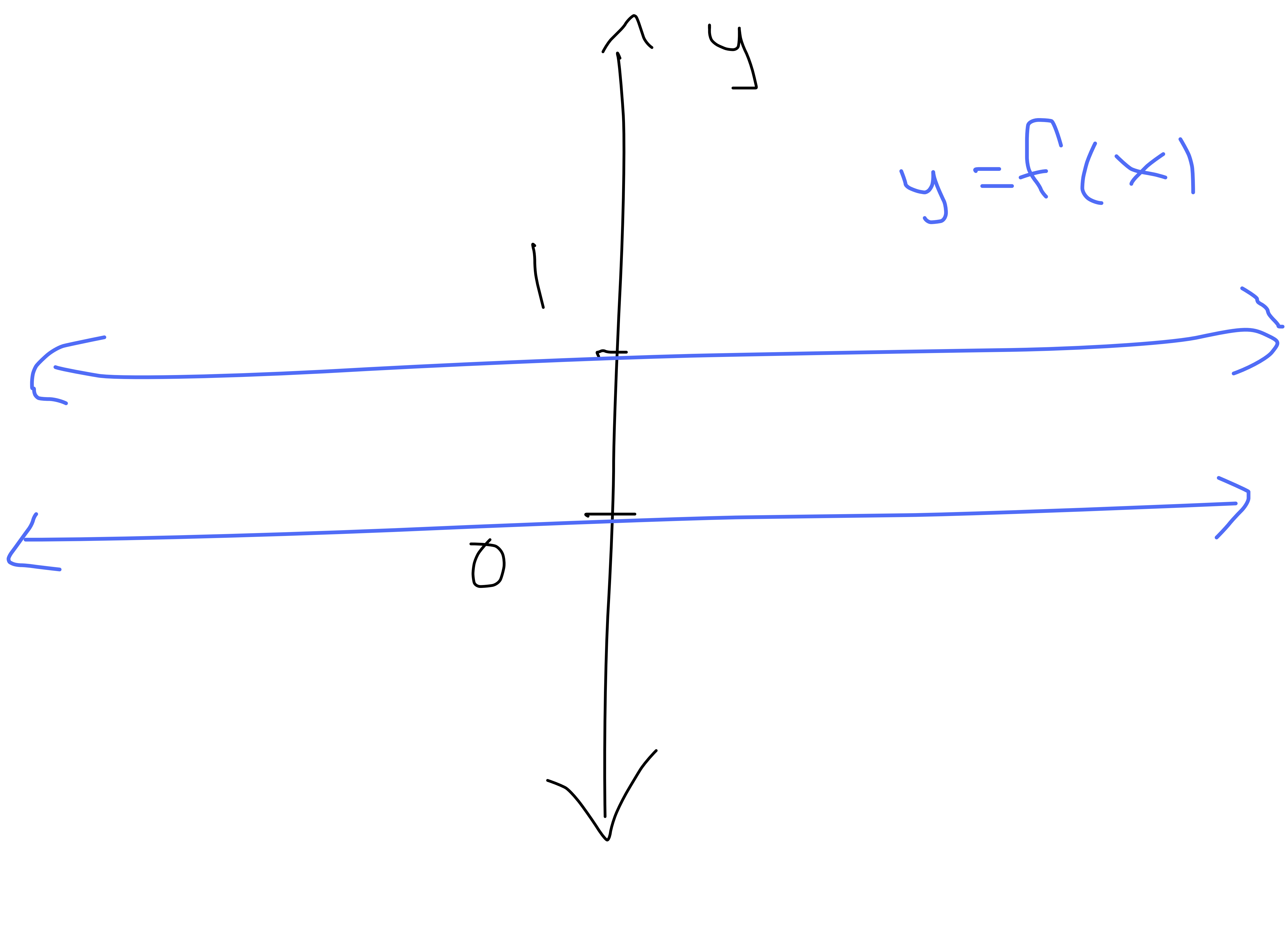
$f: \mathbb{R} \to \mathbb{R}$

$$f(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$$

What might the graph look like?

If you could graph it,

it would look like



Appears to fail the vertical line test!

This is only an appearance.

We'll talk more about this later.

Example 5: (absolute values)

$$S = T = \mathbb{R}$$

$$f : \mathbb{R} \to \mathbb{R}$$

$$f(x) = \begin{cases} x & x \geq 0 \\ -x & x \leq 0 \end{cases}$$

Usually denote by $f(x) = |x|$.

Another proof of the triangle inequality

Recall: if x, y are real numbers,

$$|x+y| \leq |x| + |y|.$$

Proof: Square $|x+y|$ to get

$$(x+y)^2 = x^2 + 2xy + y^2$$

Square $|x|+|y|$ to get

$$x^2 + y^2 + 2 \cdot |x| \cdot |y|$$

This shows

$$|x+y|^2 \le (|x|+|y|)^2$$

since we have

$$\cancel{x^2}+\cancel{y^2}+2xy \le \cancel{x^2}+\cancel{y^2}+2|x|\cdot|y|$$

same as

$$2xy \le 2|x|\cdot|y|$$

obvious.

Inequality follows from taking square roots. $\square$